OC583. Reduce the following expression to a simplified rational

$$\cos^7\frac{\pi}{9}+\cos^7\frac{5\pi}{9}+\cos^7\frac{7\pi}{9}.$$

Originally Problem 32 from the 2021 Stanford Math Tournament.

Solution 2, by the Missouri State University Problem Solving Group.

Let

$$\alpha=\cos\frac{\pi}{9},\beta=\cos\frac{5\pi}{9},\text{ and }\gamma=\cos\frac{7\pi}{9}.$$

Since

$$\cos 3\theta=4\cos^3\theta-3\cos\theta$$

and $\cos(\pi/3)=\cos(5\pi/3)=\cos(7\pi/3)=1/2$, α,β, and γ are the roots of

$$x^3-\frac{3}{4}x-\frac{1}{8}=0.$$

By Vieta's formulas, we have

$$\begin{aligned}\alpha+\beta+\gamma&=0\\ \alpha\beta+\beta\gamma+\gamma\alpha&=-\frac{3}{4}\\ \alpha\beta\gamma&=-\frac{1}{8}.\end{aligned}$$

Let

$$a_n=\alpha^n+\beta^n+\gamma^n.$$

It is well known (and easy to verify) that the a_n satisfy a recurrence relation that mirrors the polynomial of which α,β, and γ are the roots, namely

$$a_n=\frac{3}{4}a_{n-2}+\frac{1}{8}a_{n-3}.$$

The initial values of this recurrence are

$$\begin{aligned}a_0&=3,\\ a_1&=0,\\ &\text{and}\\ a_2&=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)=\frac{3}{2}.\end{aligned}$$

Using our recurrence, we find that

$$a_3 = \frac{3}{8}, \quad a_4 = \frac{9}{8}, \quad a_5 = \frac{15}{32}, \quad a_6 = \frac{57}{64}, \quad a_7 = \frac{63}{128}, \quad a_8 = \frac{93}{128}, \quad a_9 = \frac{123}{256}, \ldots$$

The answer to the question asked is therefore $a_7 = \frac{63}{128}$.